Bosnia and Herzegovina Team Selection Test 2018

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Day 1 April 21st

1 In acute triangle ABC $(AB < AC)$ let D, E and F be foots of perpedicular from A, B and C to BC, CA and AB, respectively. Let P and Q be points on line EF such that $DP \perp EF$ and $BQ = CQ$. Prove that $\angle ADP = \angle PBQ$

2 Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

3 Find all values of positive integers a and b such that it is possible to put a ones and b zeros in every of vertices in polygon with $a+b$ sides so it is possible to rotate numbers in those vertices with respect to primary position and after rotation one neighboring 0 and 1 switch places and in every other vertices other than those two numbers remain the same.

Day 2 April 22nd

4 Every square of 1000×1000 board is colored black or white. It is known that exists one square 10×10 such that all squares inside it are black and one square 10×10 such that all squares inside are white. For every square K 10×10 we define its power $m(K)$ as an absolute value of difference between number of white and black squares 1×1 in square K. Let T be a square 10×10 which has minimum power among all squares 10×10 in this board. Determine maximal possible value of $m(T)$

5 Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i.10^i$$

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The goal of Eduardo is to make M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

Proposed by Amine Natik, Morocco

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6 Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.
