30th Balkan Mathematical Olympiad
28 June - 3 July 2013, Agros - Cyprus

Contest day: Sunday 30 June 2013, 10.00 - 14.30
Place: RODON Resort, Agros

LANGUAGE: ENGLISH

Read the instructions on the separate page before you begin.
Do all problems 1 through 4. Each problem is worth 10 points.
Time allowed: 4 hours and 30 minutes

Problem 1. In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, and the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN, and let L be the projection of C onto PQ.

Show that the quadrilateral $MKLP$ is cyclic.

Problem 2. Determine all positive integers x, y and z such that

$$x^5 + 4^y = 2013^z.$$

Problem 3. Let S be the set of positive real numbers. Find all functions $f : S^3 \to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:

(a) $xf(x, y, z) = zf(z, y, x)$,

(b) $f(x, yk, k^2z) = kf(x, y, z)$,

(c) $f(1, k, k + 1) = k + 1$.

Problem 4. In a mathematical competition some competitors are friends; friendship is always mutual, that is to say that when A is a friend of B, then also B is a friend of A. We say that $n \geq 3$ different competitors A_1, A_2,..., A_n form a *weakly-friendly cycle* if A_i is not a friend of A_{i+1}, for $1 \leq i \leq n$ ($A_{n+1} = A_1$), and there are no other pairs of non-friends among the components of this cycle.

The following property is satisfied:

> for every competitor C, and every weakly-friendly cycle $\mathscr{S}$ of competitors not including C, the set of competitors D in $\mathscr{S}$ which are not friends of C has at most one element.

Prove that all competitors of this mathematical competition can be arranged into three rooms, such that every two competitors that are in the same room are friends.