Language: **Bosnian**

Subota, 28.04.2012.

Zadatak 1. Neka su A, B i C točke kružnice Γ sa centrom O, takve da je $\sphericalangle ABC > 90°$. Neka je D točka presjeka prave AB i prave okomite na AC u točki C. Neka je ℓ prava koja sadrži točku D i okomita je na pravu AO. Dalje, neka je E točka presjeka pravih ℓ i AC, a F ona točka presjeka kružnice Γ i prave ℓ koja se nalazi izmedju točaka D i E.

Dokazati da se opisane kružnice trokutova BFE i CFD dodiruju u točki F.

Zadatak 2. Dokazati da nejednakost

$$\sum_{\text{cyc}}(x+y)\sqrt{(z+x)(z+y)} \geqslant 4(xy+yz+zx)$$

važi za sve pozitivne realne brojeve x, y i z.

Suma na lijevoj strani gornje nejednakosti jednaka je

$$(x+y)\sqrt{(z+x)(z+y)}+(y+z)\sqrt{(x+y)(x+z)}+(z+x)\sqrt{(y+z)(y+x)}.$$

Zadatak 3. Za prirodan broj n neka je $P_n = \{2^n, 2^{n-1}\cdot 3, 2^{n-2}\cdot 3^2, \ldots, 3^n\}$. Za svaki podskup X skupa P_n označimo sa S_X zbir svih elemenata skupa X, pri čemu je $S_\emptyset = 0$, gdje je $\emptyset$ prazan skup. Neka je y realan broj takav da je $0 \leqslant y \leqslant 3^{n+1} - 2^{n+1}$.

Dokazati da postoji podskup Y skupa P_n za koji je $0 \leqslant y - S_Y < 2^n$.

Zadatak 4. Odrediti sve funkcije $f : \mathbb{N} \to \mathbb{N}$ koje zadovoljavaju sljedeća dva uvjeta:

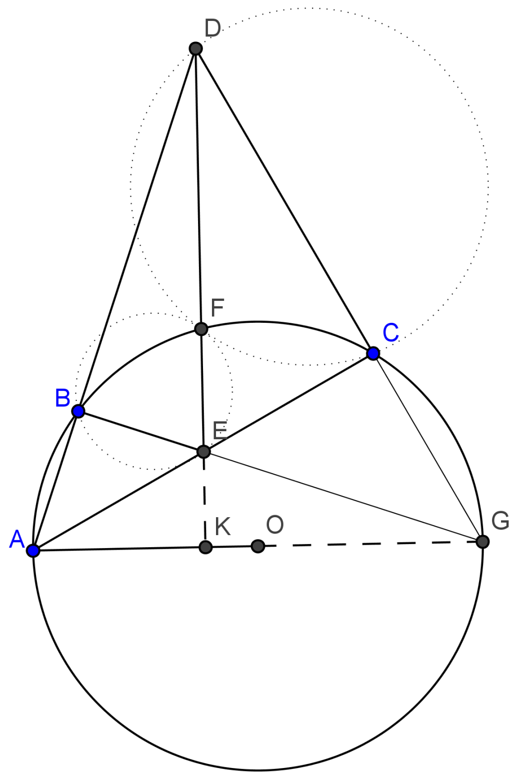
(i) $f(n!) = f(n)!$, za svaki prirodan broj n,

(ii) $m - n$ dijeli $f(m) - f(n)$, za sve različite prirodne brojeve m i n.

Svaki zadatak vrijedi 10 poena.
Vrijeme za rad: 4 sata i 30 minuta.

Problem 1.

Solution. Let $\ell \cap AO = \{K\}$ and G be the other end point of the diameter of Γ through A. Then D, C, G are collinear. Moreover, E is the orthocenter of triangle ADG. Therefore $GE \perp AD$ and G, E, B are collinear.



As $\angle CDF = \angle GDK = \angle GAC = \angle GFC$, FG is tangent to the circumcircle of triangle CFD at F. As $\angle FBE = \angle FBG = \angle FAG = \angle GFK = \angle GFE$, FG is also tangent to the circumcircle of BFE at F. Hence the circumcircles of the triangles CFD and BFE are tangent at F.

Problem 2.

Solution 1. We will obtain the inequality by adding the inequalities

$$(x+y)\sqrt{(z+x)(z+y)} \geq 2xy + yz + zx$$

for cyclic permutation of x, y, z.

Squaring both sides of this inequality we obtain

$$(x+y)^2(z+x)(z+y) \geq 4x^2y^2 + y^2z^2 + z^2x^2 + 4xy^2z + 4x^2yz + 2xyz^2$$

which is equivalent to

$$x^3y + xy^3 + z(x^3 + y^3) \geq 2x^2y^2 + xyz(x+y)$$

which can be rearranged to

$$(xy + yz + zx)(x-y)^2 \geq 0,$$

which is clearly true.

Solution 2. For positive real numbers x, y, z there exists a triangle with the side lengths $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ and the area $K = \sqrt{xy+yz+zx}/2$.

The existence of the triangle is clear by simple checking of the triangle inequality. To prove the area formula, we have

$$K = \frac{1}{2}\sqrt{x+y}\sqrt{z+x}\sin\alpha,$$

where α is the angle between the sides of length $\sqrt{x+y}$ and $\sqrt{z+x}$. On the other hand, from the law of cosines we have

$$\cos\alpha = \frac{x+y+z+x-y-z}{2\sqrt{(x+y)(z+x)}} = \frac{x}{\sqrt{(x+y)(z+x)}}$$

and

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\sqrt{xy+yz+zx}}{\sqrt{(x+y)(z+x)}}.$$

Now the inequality is equivalent to

$$\sqrt{x+y}\sqrt{y+z}\sqrt{z+x}\sum_{cyc}\sqrt{x+y} \geq 16K^2.$$

This can be rewritten as

$$\frac{\sqrt{x+y}\sqrt{y+z}\sqrt{z+x}}{4K} \geq 2\frac{K}{\sum_{cyc}\sqrt{x+y}/2}$$

to become the Euler inequality $R \geq 2r$.

Problem 3.

Solution 1. Let $\alpha = 3/2$ so $1 + \alpha > \alpha^2$.

Given y, we construct Y algorithmically. Let $Y = \varnothing$ and of course $S_\varnothing = 0$. For $i = 0$ to m, perform the following operation:

$$\text{If } S_Y + 2^i 3^{m-i} \le y, \text{ then replace } Y \text{ by } Y \cup \{2^i 3^{m-i}\}.$$

When this process is finished, we have a subset Y of P_m such that $S_Y \le y$.

Notice that the elements of P_m are in ascending order of size as given, and may alternatively be described as $2^m, 2^m\alpha, 2^m\alpha^2, \ldots, 2^m\alpha^m$. If any member of this list is not in Y, then no two consecutive members of the list to the left of the omitted member can both be in Y. This is because $1 + \alpha > \alpha^2$, and the greedy nature of the process used to construct Y.

Therefore either $Y = P_m$, in which case $y = 3^{m+1} - 2^{m+1}$ and all is well, or at least one of the two leftmost elements of the list is omitted from Y.

If 2^m is not omitted from Y, then the algorithmic process ensures that $(S_Y - 2^m) + 2^{m-1}3 > y$, and so $y - S_Y < 2^m$. On the other hand, if 2^m is omitted from Y, then $y - S_Y < 2^m$).

Solution 2. Note that $3^{m+1} - 2^{m+1} = (3-2)(3^m + 3^{m-1} \cdot 2 + \cdots + 3 \cdot 2^{m-1} + 2^m) = S_{P_m}$. Dividing every element of P_m by 2^m gives us the following equivalent problem:

Let m be a positive integer, $a = 3/2$, and $Q_m = \{1, a, a^2, \ldots, a^m\}$. Show that for any real number x satisfying $0 \le x \le 1 + a + a^2 + \cdots + a^m$, there exists a subset X of Q_m such that $0 \le x - S_X < 1$.

We will prove this problem by induction on m. When $m = 1$, $S_\varnothing = 0$, $S_{\{1\}} = 1$, $S_{\{a\}} = 3/2$, $S_{\{1,a\}} = 5/2$. Since the difference between any two consecutive of them is at most 1, the claim is true.

Suppose that the statement is true for positive integer m. Let x be a real number with $0 \le x \le 1 + a + a^2 + \cdots + a^{m+1}$. If $0 \le x \le 1 + a + a^2 + \cdots + a^m$, then by the induction hypothesis there exists a subset X of $Q_m \subset Q_{m+1}$ such that $0 \le x - S_X < 1$.

If $\dfrac{a^{m+1} - 1}{a - 1} = 1 + a + a^2 + \cdots + a^m < x$, then $x > a^{m+1}$ as

$$\frac{a^{m+1} - 1}{a - 1} = 2(a^{m+1} - 1) = a^{m+1} + (a^{m+1} - 2) \ge a^{m+1} + a^2 - 2 = a^{m+1} + \frac{1}{4}.$$

Therefore $0 < (x - a^{m+1}) \le 1 + a + a^2 + \cdots + a^m$. Again by the induction hypothesis, there exists a subset X of Q_m satisfying $0 \le (x - a^{m+1}) - S_X < 1$. Hence $0 \le x - S_{X'} < 1$ where $X' = X \cup \{a^{m+1}\} \subset Q_{m+1}$.

Problem 4.

Solution 1. There are three such functions: the constant functions 1, 2 and the identity function $\mathrm{id}_{\mathbf{Z}^+}$. These functions clearly satisfy the conditions in the hypothesis. Let us prove that there are only ones.

Consider such a function f and suppose that it has a fixed point $a \geq 3$, that is $f(a) = a$. Then $a!, (a!)!, \cdots$ are all fixed points of f, hence the function f has a strictly increasing sequence $a_1 < a_2 < \cdots < a_k < \cdots$ of fixed points. For a positive integer n, $a_k - n$ divides $a_k - f(n) = f(a_k) - f(n)$ for every $k \in \mathbf{Z}^+$. Also $a_k - n$ divides $a_k - n$, so it divides $a_k - f(n) - (a_k - n) = n - f(n)$. This is possible only if $f(n) = n$, hence in this case we get $f = \mathrm{id}_{\mathbf{Z}^+}$.

Now suppose that f has no fixed points greater than 2. Let $p \geq 5$ be a prime and notice that by Wilson's Theorem we have $(p-2)! \equiv 1 \pmod{p}$. Therefore p divides $(p-2)! - 1$. But $(p-2)! - 1$ divides $f((p-2)!) - f(1)$, hence p divides $f((p-2)!) - f(1) = (f(p-2))! - f(1)$. Clearly we have $f(1) = 1$ or $f(1) = 2$. As $p \geq 5$, the fact that p divides $(f(p-2))! - f(1)$ implies that $f(p-2) < p$. It is easy to check, again by Wilson's Theorem, that p does not divide $(p-1)! - 1$ and $(p-1)! - 2$, hence we deduce that $f(p-2) \leq p-2$. On the other hand, $p - 3 = (p-2) - 1$ divides $f(p-2) - f(1) \leq (p-2) - 1$. Thus either $f(p-2) = f(1)$ or $f(p-2) = p-2$. As $p - 2 \geq 3$, the last case is excluded, since the function f has no fixed points greater than 2. It follows $f(p-2) = f(1)$ and this property holds for all primes $p \geq 5$. Taking n any positive integer, we deduce that $p - 2 - n$ divides $f(p-2) - f(n) = f(1) - f(n)$ for all primes $p \geq 5$. Thus $f(n) = f(1)$, hence f is the constant function 1 or 2.

Solution 2. Note first that if $f(n_0) = n_0$, then $m - n_0 | f(m) - m$ for all $m \in \mathbf{Z}^+$. If $f(n_0) = n_0$ for infinitely many $n_0 \in \mathbf{Z}^+$, then $f(m) - m$ has infinitely many divisors, hence $f(m) = m$ for all $m \in \mathbf{Z}^+$. On the other hand, if $f(n_0) = n_0$ for some $n_0 \geq 3$, then f fixes each term of the sequence $(n_k)_{k=0}^{\infty}$, which is recursively defined by $n_k = n_{k-1}!$. Hence if $f(3) = 3$, then $f(n) = n$ for all $n \in \mathbf{Z}^+$.

We may assume that $f(3) \neq 3$. Since $f(1) = f(1)!$, and $f(2) = f(2)!$, $f(1), f(2) \in \{1, 2\}$. We have $4 = 3! - 2 | f(3)! - f(2)$. This together with $f(3) \neq 3$ implies that $f(3) \in \{1, 2\}$. Let $n > 3$, then $n! - 3 | f(n)! - f(3)$ and $3 \nmid f(n)!$, i.e. $f(n)! \in \{1, 2\}$. Hence we conclude that $f(n) \in \{1, 2\}$ for all $n \in \mathbf{Z}^+$.

If f is not constant, then there exist positive integers m, n with $\{f(n), f(m)\} = \{1, 2\}$. Let $k = 2 + \max\{m, n\}$. If $f(k) \neq f(m)$, then $k - m | f(k) - f(m)$. This is a contradiction as $|f(k) - f(m)| = 1$ and $k - m \geq 2$.

Therefore the functions satisfying the conditions are $f \equiv 1, f \equiv 2, f = \mathrm{id}_{\mathbf{Z}^+}$.

OFFICIAL RESULTS OF BMO 2012

Rank	Code	Name	Surname	P1	P2	P3	P4	∑	Medal
1	BIH3	Harun	HINDIJA	10	10	10	10	40	GOLD
1	HEL5	Alexandros	MOUSATOV	10	10	10	10	40	GOLD
1	ROU3	Octav	DRĂGOI	10	10	10	10	40	GOLD
1	SRB1	Teodor von	BURG	10	10	10	10	40	GOLD
1	SRB3	Dušan	ŠOBOT	10	10	10	10	40	GOLD
1	SRB4	Igor	SPASOJEVIĆ	10	10	10	10	40	GOLD
1	TUR2	Yunus Emre	DEMİRCİ	10	10	10	10	40	GOLD
1	TUR4	Mehmet	SÖNMEZ	10	10	10	10	40	GOLD
1	TUR5	Berfin	ŞİMŞEK	10	10	10	10	40	GOLD
10	HEL3	Panagiotis	LOLAS	10	10	10	9	39	GOLD
10	ROU2	Ömer	CERRAHOĞLU	10	10	9	10	39	GOLD
10	ROU5	Ştefan	SPĂTARU	10	10	10	9	39	GOLD
10	TUR3	Ufuk	KANAT	10	10	9	10	39	GOLD
10	TURB4	Muhammed İkbal	ULVİ	10	10	9	10	39	GOLD
15	BGR5	Yordan	YORDANOV	10	10	9	9	38	SILVER
15	SAU6	Al Yazeed	PASUNI	10	10	8	10	38	SILVER
15	SRB5	Rade	ŠPEGAR	10	10	10	8	38	SILVER
15	TURB5	Burak	VARICI	10	10	10	8	38	SILVER
19	ROU1	Hai	TRAN BACH	10	10	10	7	37	SILVER
20	TUR6	Mehmet Akif	YILDIZ	10	10	6	10	36	SILVER
21	HEL6	Zacharias	TSAMPASIDIS	10	10	10	4	34	SILVER
22	FRA1	Arthur	BLANC-RENAUDIE	10	10	10	3	33	SILVER
22	ROU4	Ştefan	IVANOVICI	10	10	9	4	33	SILVER
22	TURB2	Eren	DURLANIK	10	10	9	4	33	SILVER
22	UNK5	Matei	MANDACHE	10	10	9	4	33	SILVER
26	BGR1	Ivailo	HARTARSKI	10	10	9	3	32	SILVER
26	ITA1	Gioacchino	ANTONELLI	10	10	9	3	32	SILVER
26	TURB6	Orhan Tahir	YAVAŞCAN	10	10	10	2	32	SILVER
29	BGR6	Bogdan	STANKOV	10	9	8	4	31	SILVER
29	BIH2	Ratko	DARDA	10	10	1	10	31	SILVER
29	IDN2	Christa Lorenzia	SOESANTO	10	10	10	1	31	SILVER
29	ITA2	Dario	ASCARI	10	10	10	1	31	SILVER
29	TUR1	Mehmet Efe	AKENGİN	10	10	1	10	31	SILVER
29	TURB3	Fatih	KALEOĞLU	10	10	10	1	31	SILVER
35	FRA4	Seginus	MOWLAVI	10	10	10	0	30	SILVER
35	FRA5	Matthieu	PIQUEREZ	10	7	10	3	30	SILVER
35	ITA3	Luigi	PAGANO	10	10	9	1	30	SILVER
35	ITA5	Daniele	SEMOLA	10	10	10	0	30	SILVER
35	KAZ1	Saken	DUSHAYEV	10	10	10	0	30	SILVER
35	KAZ5	Alibek	SAILANBAYEV	10	10	0	10	30	SILVER
35	TKM1	Palvan	AGAMYRADOV	10	10	10	0	30	SILVER
35	UNK3	Daniel	HU	10	0	10	10	30	SILVER
43	BIH4	Hamza	MERZIC	10	10	0	8	28	BRONZE
43	ITA6	Marco	TREVISIOL	5	10	9	4	28	BRONZE
45	KAZ3	Sergazy	KALMURZAYEV	10	10	1	6	27	BRONZE
45	TJK1	Muhammadfiruz	HASANOV	6	10	10	1	27	BRONZE
45	UNK1	Robin	ELLIOTT	8	10	8	1	27	BRONZE

48	MDA5	Andrei	IVANOV	10	10	5	0	25	BRONZE
48	ROU6	Florina	TOMA	10	10	4	1	25	BRONZE
50	FRA2	Sébastien	CHEVALEYRE	6	10	8	0	24	BRONZE
50	KAZ4	Yerniyaz	NURGABYLOV	10	4	0	10	24	BRONZE
50	TJK6	Saidmuhammadi	SHODAVLAT	10	10	1	3	24	BRONZE
53	BGR4	Pavlena	NENOVA	10	10	3	0	23	BRONZE
53	FRA6	Victor	QUACH	4	10	9	0	23	BRONZE
53	KAZ6	Temirlan	SEILOV	10	10	0	3	23	BRONZE
53	MDA2	Cristian	CERNEANU	10	3	10	0	23	BRONZE
53	MKD4	Filip	STANKOVSKI	10	10	1	2	23	BRONZE
53	TJK3	Elmurod	KHUSRAVI	10	10	0	3	23	BRONZE
53	TURB1	Eray	AYDIN	10	10	0	3	23	BRONZE
60	BGR2	Radoslav	KAFOV	10	10	1	1	22	BRONZE
60	IDN1	Bivan Alzacky	HARMANTO	10	10	1	1	22	BRONZE
60	KAZ2	Akzhol	IBRAIMOV	10	10	2	0	22	BRONZE
60	MDA3	Mihai	INDRICEAN	10	10	1	1	22	BRONZE
60	TJK4	Musokhon	MUQAYUMKHONOV	10	10	1	1	22	BRONZE
65	FRA3	Louise	GASSOT	10	0	10	1	21	BRONZE
65	ITA4	Francesco	SALA	10	10	0	1	21	BRONZE
65	MDA1	Cristian	ZANOCI	10	10	1	0	21	BRONZE
65	MDA4	Dinis	CHEIAN	10	10	1	0	21	BRONZE
65	MDA6	Dionisie	NIPOMICI	10	10	1	0	21	BRONZE
65	MKD1	Rosica	DEJANOVSKA	10	10	1	0	21	BRONZE
65	SRB2	Ivan	DAMNJANOVIĆ	0	10	10	1	21	BRONZE
65	TJK2	Abdulvosit	ISMAILOV	10	10	0	1	21	BRONZE
65	TJK5	Sadi	OKILNAZARZODA	10	10	0	1	21	BRONZE
65	TKM5	Ashyrmuhammet	SERDAROV	10	10	1	0	21	BRONZE
75	ALB3	Erjona	TOPALLI	10	10	0	0	20	BRONZE
75	AZE1	Izmir	ALIYEV	10	10	0	0	20	BRONZE
75	CYP5	Thomas	THOMA	10	10	0	0	20	BRONZE
75	HEL1	Panagiotis	DIMAKIS	10	10	0	0	20	BRONZE
75	MKD2	Vasil	KUZEVSKI	10	10	0	0	20	BRONZE
75	MKD5	Stefan	STOJCHEVSKI	10	9	1	0	20	BRONZE
75	SAU1	Saleh	ALGAMDI	10	10	0	0	20	BRONZE
75	SAU2	Abdulrahman	ALHARBI	10	10	0	0	20	BRONZE
75	SAU3	Doha	ALJEDDAWI	10	10	0	0	20	BRONZE
75	SAU4	Hasan	EID	10	10	0	0	20	BRONZE
75	SRB6	Ivan	TANASIJEVIĆ	10	10	0	0	20	BRONZE
75	TKM2	Shanur	ALLABAYEV	10	9	0	1	20	BRONZE
75	UNK2	Gabriel	GENDLER	1	9	10	0	20	BRONZE
88	CYP4	Anastasios	STYLIANOU	8	10	0	1	19	HM
89	BIH1	Vladimir	IVKOVIC	10	4	0	4	18	HM
90	BGR3	Radoslav	KOMITOV	10	6	0	0	16	HM
91	CYP6	Marios	VOSKOU	0	10	4	1	15	HM
91	HEL2	Iason	KOUNTOURIDIS	4	10	1	0	15	HM
91	IDN5	Christopher	WIRIAWAN	4	10	0	1	15	HM
91	TKM4	Nazar	RAHMANOV	10	4	1	0	15	HM
91	UNK4	Matthew	JASPER	4	0	10	1	15	HM
96	AZE3	Suleyman	HASANZADE	10	4	0	0	14	HM
96	AZE6	Gunduz	SALIMLI	10	4	0	0	14	HM

96	MKD3	Aleksandar	MOMIROVSKI	4	10	0	0	14	HM
96	UNK6	Harry	METREBIAN	10	4	0	0	14	HM
100	MNE2	Olja	KRSTOVIĆ	0	10	2	1	13	HM
100	SAU5	Husain	EID	9	4	0	0	13	
102	AZE2	Hasan	HASANZADE	10	1	0	1	12	HM
102	AZE4	Shahin	HUSEYNLI	10	2	0	0	12	HM
102	BIH5	Sead	DELALIC	2	10	0	0	12	HM
102	CYP2	Georgios-Rafael	PATSALIDIS	10	0	1	1	12	HM
106	ALB6	Genti	GJIKA	1	10	0	0	11	HM
106	AZE5	Farhad	SAFARLI	10	1	0	0	11	HM
106	BIH6	Ivan	BARTULOVIC	1	10	0	0	11	HM
106	MKD6	Marija	TEPEGJOZOVA	3	8	0	0	11	
106	TKM3	Fahruddin	BARATOV	1	10	0	0	11	HM
111	TKM6	Merdan	SOLTANOV	10	0	0	0	10	HM
112	CYP3	Andreas	SOTIRIOU	4	5	0	0	9	
113	ALB4	Gledis	KALLÇO	2	4	1	0	7	
114	HEL4	Eleftherios	MALLIOS	6	0	0	0	6	
115	CYP1	Dimitris	MOUYIS	0	3	1	0	4	
116	MNE1	Oleg	CMILJANIĆ	1	2	0	0	3	
117	ALB5	Florida	AHMETAJ	1	0	0	1	2	
118	ALB1	Boriana	GJURA	0	1	0	0	1	
119	AFG1	Abdul Qayoum	SAMADI	0	0	0	0	0	
119	AFG2	Mohammad Omar	HAMIDZAI	0	0	0	0	0	
119	AFG3	Sayed Edris	HOSAINI	0	0	0	0	0	
119	AFG4	Ali Homayoun	SADEGZADA	0	0	0	0	0	
119	ALB2	Fatjon	GERA	0	0	0	0	0	