Balkan MO 2011

1 Let $ABCD$ be a cyclic quadrilateral which is not a trapezoid and whose diagonals meet at E. The midpoints of AB and CD are F and G respectively, and ℓ is the line through G parallel to AB. The feet of the perpendiculars from E onto the lines ℓ and CD are H and K, respectively. Prove that the lines EF and HK are perpendicular.

2 Given real numbers x, y, z such that $x + y + z = 0$, show that

$$\frac{x(x+2)}{2x^2+1} + \frac{y(y+2)}{2y^2+1} + \frac{z(z+2)}{2z^2+1} \geq 0$$

When does equality hold?

3 Let S be a finite set of positive integers which has the following property:if x is a member of S,then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ratio y/x is a power of a prime number. A non-empty subset T of S is *bad* if whenever $x, y \in T$ and $x < y$, the ratio y/x is not a power of a prime number. A set of an element is considered both *good* and *bad*. Let k be the largest possible size of a *good* subset of S. Prove that k is also the smallest number of pairwise-disjoint *bad* subsets whose union is S.

4 Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is at least $3/2$.