Bosnia Herzegovina Team Selection Test 2012

Day 1

1 Let D be the midpoint of the arc $B-A-C$ of the circumcircle of $\triangle ABC (AB < AC)$. Let E be the foot of perpendicular from D to AC. Prove that $|CE| = \frac{|BA|+|AC|}{2}$.

2 Prove for all positive real numbers a, b, c, such that $a^2 + b^2 + c^2 = 1$:

$$\frac{a^3}{b^2+c} + \frac{b^3}{c^2+a} + \frac{c^3}{a^2+b} \geq \frac{\sqrt{3}}{1+\sqrt{3}}.$$

3 Prove that for all odd prime numbers p there exist a natural number $m < p$ and integers x_1, x_2, x_3 such that:

$$mp = x_1^2 + x_2^2 + x_3^2.$$

Day 2

4 Define a function $f : \mathbb{N} \to \mathbb{N}$,

$$f(1) = p + 1,$$

$$f(n+1) = f(1) \cdot f(2) \cdots f(n) + p,$$

where p is a prime number. Find all p such that there exists a natural number k such that $f(k)$ is a perfect square.

5 Given is a triangle $\triangle ABC$ and points M and K on lines AB and CB such that $AM = AC = CK$. Prove that the length of the radius of the circumcircle of triangle $\triangle BKM$ is equal to the lenght OI, where O and I are centers of the circumcircle and the incircle of $\triangle ABC$, respectively. Also prove that $OI \perp MK$.

6 A unit square is divided into polygons, so that all sides of a polygon are parallel to sides of the given square. If the total length of the segments inside the square (without the square) is $2n$ (where n is a positive real number), prove that there exists a polygon whose area is greater than $\frac{1}{(n+1)^2}$.